THE REDUCTION OF MULTIPLE L-INTEGRALS OF SEPARATED FUNCTIONS TO ITERATED L-INTEGRALS*

BY

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Introduction

In the second volume of his book, *Lectures on the Theory of Functions of Real Variables*,† Professor Pierpont defines an integral which he calls an L-integral, and develops a theory of integration which contains Lebesgue's theory as a special case. Up to a certain point, in Pierpont's theory, the field of integration does not need to be measurable, nor does the integrand need to be a measurable function in the sense of Lebesgue. In case the field of integration is measurable Pierpont shows that "an L-integrable function is integrable in Lebesgue's sense, and conversely; moreover, both have the same value."

In the Pierpont theory it is only in the theorems relating to the reduction of multiple L-integrals to iterated L-integrals, and in the theorems leading up to this reduction, that the condition that the field be measurable is introduced. It is true that no one has yet defined a very satisfactory non-measurable set and yet the existence of such sets has been established both by Lebesgue‡ and by Professor Van Vleck.§ The latter has also derived some of the properties which non-measurable sets must possess.

In this paper I shall give sufficient conditions for the existence and equality of the multiple and iterated L-integrals for a certain class of functions, called separated functions, defined over a field which may not be measurable. If the function is equal to or greater than some arbitrarily small positive quantity, these conditions are also necessary. In case the field is measurable a separated function defined over it becomes a measurable function and my extra conditions are necessarily satisfied.

In what follows it is understood that all functions and all fields of integra-

* Presented to the Society, December 31, 1913.

† Published by Ginn & Co., Boston, 1912. These will be referred to as *Lectures*, II.

‡ Bulletin de la Société Mathématique de France, vol. 35 (1907), pp. 202–212.

§ *On a non-measurable set of points with an example.* These Transactions, vol. 9 (1908), pp. 237–244.

tion are limited, and that the integral signs used denote L-integrals.* In case a Riemann integral is used the sign of integration will be prefixed thus: $R\int$.

1. Preliminary Definitions†

Let $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ be a finite or enumerably infinite set of point aggregates in n-way space which we denote by $\mathfrak{R}_n$. The point set formed by the points which belong to at least one of the sets $\mathfrak{A}_n$ is called the *union* of these sets, and is denoted by $U\{\mathfrak{A}_1, \mathfrak{A}_2, \cdots\}$ or by $U\{\mathfrak{A}_n\}$. The point set formed by the points which are common to all the sets $\mathfrak{A}_n$ is called the *divisor* of these sets, and is denoted by $Dv\{\mathfrak{A}_1, \mathfrak{A}_2, \cdots\}$ or by $Dv\{\mathfrak{A}_n\}$.

Let $\mathfrak{A}$ be a point set in $\mathfrak{R}_n$. Let $D = \{d_\iota\}$ be an enumerable set of metric sets. If each point of $\mathfrak{A}$ lies in at least one of the sets d_ι, D is called an *enclosure* of $\mathfrak{A}$. The sets d_ι are called *cells*. If the cells d_ι are measurable, D is called a *measurable enclosure* of $\mathfrak{A}$.

Let $\mathfrak{A} = U\{\mathfrak{A}_1, \mathfrak{A}_2\}$. If there exists a measurable enclosure A_1 of $\mathfrak{A}_1$ and a measurable enclosure A_2 of $\mathfrak{A}_2$, such that $Dv\{A_1, A_2\}$ is a null set, that is of measure zero, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are said to be *separated sets*.

If $\mathfrak{A}_1 < \mathfrak{A}$, and $\mathfrak{A}_1$ and $\mathfrak{A} - \mathfrak{A}_1$ are separated sets, $\mathfrak{A}_1$ is said to be a *separated part* of $\mathfrak{A}$.

If $\mathfrak{A} = U\{\mathfrak{A}_n\}$ and each pair of sets $\mathfrak{A}_i$, $\mathfrak{A}_j$ are separated sets, the sets $\{\mathfrak{A}_n\}$ are said to form a *separated division* of $\mathfrak{A}$. It is evident that each $\mathfrak{A}_n$ is a separated part of $\mathfrak{A}$.

Let $f(x_1, \cdots x_n)$ be defined at the points of $\mathfrak{A}$. Let $\alpha \geqq 0$ and $\beta \geqq 0$ be two numbers chosen at pleasure. If, for every pair of such numbers, the sets of points at which

$$f \leqq -\alpha, \qquad \beta \leqq f \tag{1}$$

are separated parts of $\mathfrak{A}$, f is said to be a *separated function* in $\mathfrak{A}$. In what follows the two sets defined by relations (1) are denoted by $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ respectively.

Let $n = r + s$. If we set the coördinates $x_{r+1}, \cdots x_n$, of the points of $\mathfrak{A}$, all equal to zero, we get a point set $\mathfrak{B}$ in r-way space $\mathfrak{R}_r$. We call $\mathfrak{B}$ the *projection* of $\mathfrak{A}$ on $\mathfrak{R}_r$. Let x be an arbitrary but fixed point of $\mathfrak{B}$ with coördinates $(x_1, \cdots x_r)$. Consider all the points of $\mathfrak{A}$ whose first r coördinates are the coördinates of x. We denote these points by $\mathfrak{C}(x)$, and, when no confusion can arise, simply by $\mathfrak{C}$. We call $\mathfrak{C}(x)$ the section of $\mathfrak{A}$ corresponding to x, and $\mathfrak{C}$ and $\mathfrak{B}$ *components* of $\mathfrak{A}$. It is convenient to write $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$.

Let x $(x_1, \cdots x_n)$ be an arbitrary but fixed point of $\mathfrak{A}$. Consider the

* Defined later in § 1.

† These definitions are, for the most part, given in *Lectures*, II.

points in $(n + 1)$-way space whose first n coördinates are those of x. They are represented by $(x_1, \cdots x_n, x_{n+1})$, where x_{n+1} ranges from $-\infty$ to $+\infty$. We call this set of points an *ordinate* through x. If x_{n+1} is restricted so that

$$0 \leqq x_{n+1} \leqq l,$$

we call the resulting set of points a positive *ordinate of length* l. In case $l = 0$ the ordinate through x consists of the single point x itself.

Let $f(x_1, \cdots x_n) \geqq 0$ be defined over $\mathfrak{A}$. Through each point x of $\mathfrak{A}$ let an ordinate be erected of length equal to $f(x)$. The resulting set of points in $\mathfrak{R}_{n+1}$ we call the *integrand set* $\mathfrak{J}$ of f.*

Let $f(x_1, \cdots x_n)$ be defined over $\mathfrak{A}$ and such that

$$0 \leqq f < l.$$

Through each point of $\mathfrak{A}$ let an ordinate be erected of constant length l. The resulting set we denote by S; from its definition $S > \mathfrak{J}$. We call S an *enclosing ordinate set* of $\mathfrak{J}$.

If $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is such that†

$$\int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}} = \overline{\overline{\mathfrak{A}}},$$

then $\mathfrak{A}$ is said to be *L-iterable with respect to* $\mathfrak{B}$.

The points $x(x_1, \cdots x_n)$ such that

$$a_1 \leqq x_1 \leqq b_1, \quad \cdots \quad a_n \leqq x_n \leqq b_n$$

form what is called a *standard rectangular cell.* An enclosure of $\mathfrak{A}$ of non-overlapping standard cells is called a *standard enclosure* of $\mathfrak{A}$. Let E be a standard enclosure of cells e_n. If

$$\sum \overline{e}_n - \overline{\overline{\mathfrak{A}}} < \epsilon,$$

E is called a *standard ϵ-enclosure* of $\mathfrak{A}$. It has been proved that for each $\epsilon > 0$ there are standard ϵ-enclosures of any limited $\mathfrak{A}$.‡

Let $\epsilon_1 > \epsilon_2 > \epsilon_3 \cdots \doteq 0$. Let E_n be a sequence of standard ϵ_n-enclosures of $\mathfrak{A}$ such that the cells of E_{n+1} lie in E_n. The set

$$\mathfrak{A}_e = Dv\{E_n\}$$

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* An integrand set for a function of both signs can readily be defined, but such a set is not needed in what follows.

† This notation is used to denote the upper measure of a set of points, viz., $\overline{\overline{\mathfrak{A}}}$ is read upper measure of $\mathfrak{A}$. The symbols $\overline{\text{meas.}}$ and meas. are also used before parentheses to denote upper measure and measure.

If two sets $\mathfrak{A}_1$ and $\mathfrak{A}_2$, in $\mathfrak{R}_n$ and $\mathfrak{R}_m$ respectively, enter into the same consideration, the symbol $\overline{\overline{\mathfrak{A}}}_1$ is understood to mean n dimensional upper measure and $\overline{\overline{\mathfrak{A}}}_2$ to mean m dimensional upper measure.

A definition of the integral here used is given below.

‡ *Lectures*, II, § 363, 2.

is called an outer associated set of $\mathfrak{A}$, and it has been proved that $\mathfrak{A}_e$ is measurable and meas. $\mathfrak{A}_e = \overline{\mathfrak{A}}$.*

The following is the Pierpont definition of an L-integral. Let $f(x_1, \cdots x_n)$ be defined over $\mathfrak{A}$. Let D be a division of $\mathfrak{A}$ into separated cells $d_1, d_2, \cdots d_n \cdots$ of norm d. Let†

$$M_\iota = \text{Max} f \text{ in } d_\iota \quad \text{and} \quad m_\iota = \text{Min} f \text{ in } d_\iota .$$

Then

$$\overline{S}_D = \sum M_\iota \overline{\overline{d}}_\iota \quad \text{and} \quad \underline{S}_D = \sum m_\iota \overline{\overline{d}}_\iota$$

are called the upper and lower sums of f over $\mathfrak{A}$ for the division D. Since f is limited in $\mathfrak{A}$, Max $\underline{S}_D$ and Min $\overline{S}_D$, with respect to all finite and enumerably infinite separated divisions of $\mathfrak{A}$, are finite. We call them respectively the *lower and upper L-integrals* of f over $\mathfrak{A}$ and write

$$\text{Max } \underline{S}_D = \underline{\int}_{\mathfrak{A}} f, \qquad \text{Min } \overline{S}_D = \overline{\int}_{\mathfrak{A}} f.$$

In case these two have a common value we denote it by

$$\int_{\mathfrak{A}} f,$$

which we call the *L-integral* of f over $\mathfrak{A}$.

2. Separated Sets and Separated Functions

Theorem 1. *Let A_ι, $\iota = 1, 2, 3 \cdots$ be separated parts of $\mathfrak{A}$. Then $\mathfrak{D} = Dv\{A_\iota\}$ is also a separated part of $\mathfrak{A}$.*

Since A_ι is a separated part of $\mathfrak{A}$, there exist measurable enclosures, D_ι and E_ι, of A_ι and $\mathfrak{A} - A_\iota$ respectively, such that $Dv\{D_\iota, E_\iota\}$ is a null set. $\mathfrak{D} \leqq Dv\{D_\iota\}$ which is measurable by *Lectures*, II, § 361. Also,

$$\mathfrak{A} - \mathfrak{D} \leqq U\{E_\iota\},$$

which is measurable by *Lectures*, II, § 359. Any point common to $Dv\{D_\iota\}$ and $U\{E_\iota\}$ lies in some $Dv\{D_\kappa, E_\kappa\}$. But we have seen that all such sets are null sets and consequently their union is a null set. We have thus enclosed $\mathfrak{D}$ and $\mathfrak{A} - \mathfrak{D}$ in measurable enclosures whose common points form a null set, and hence, by definition, $\mathfrak{D}$ and $\mathfrak{A} - \mathfrak{D}$ are separated parts of $\mathfrak{A}$.

Theorem 2. *Let A_ι, $\iota = 1, 2, 3 \cdots$ be separated parts of $\mathfrak{A}$. Then $U\{A_\iota\}$ is also a separated part of $\mathfrak{A}$.*

Let $\mathfrak{A} - A_\iota = C_\iota$. Since the sets A_ι are separated parts of $\mathfrak{A}$, so are the

* *Lectures*, II, § 369.

† The notation Max f is Pierpont's; it means the least upper bound of f. Similarly for Min f.

sets C_ι and consequently, by Theorem 1, $Dv\{C_\iota\}$ is a separated part of $\mathfrak{A}$. Hence $U\{A_\iota\} = \mathfrak{A} - Dv\{C_\iota\}$ is a separated part of $\mathfrak{A}$.

THEOREM 3. *Let $f(x_1, \cdots x_n)$ be a separated function defined over $\mathfrak{A}$. Let M and N be two numbers chosen at pleasure, but so that $M \leqq N$. Then the set of points at which*

$$M \leqq f \leqq N$$

forms a separated part of $\mathfrak{A}$.

Case 1. $M = N$. To fix ideas let $M = N = \kappa > 0$. Let

$$\beta_1 < \beta_2 < \beta_3 \cdots \doteq \kappa$$

and

$$\beta_1' > \beta_2' > \beta_3' \cdots \doteq \kappa.$$

Then $Dv\{\mathfrak{A}_{\beta_\iota}, \mathfrak{A}_{\beta_\iota'}\}$ is the set for which $f = \kappa$, and by Theorem 1 this is a separated part of $\mathfrak{A}$.

Case 2. $0 \leqq M < N$. The set for which $M \leqq f < N$ is $Dv\{\mathfrak{A}_{\beta_1}, \mathfrak{A} - \mathfrak{A}_{\beta_2}\}$, $\beta_1 = M$, $\beta_2 = N$. This is a separated part of $\mathfrak{A}$ by Theorem 1. By case 1, the points at which $f = N$ form a separated part of $\mathfrak{A}$. The present case is then proved by Theorem 2.

Case 3. $M < N \leqq 0$. Substitute α for β in proof of case 2.

Case 4. $M < 0 < N$. The set for which $M < f < N$ is

$$Dv\{\mathfrak{A} - \mathfrak{A}_\beta, \mathfrak{A} - \mathfrak{A}_\alpha\}, \qquad \alpha = M, \quad \beta = N,$$

which is a separated part of $\mathfrak{A}$ by Theorem 1. The remainder of the proof of this case is similar to that of case 2.

THEOREM 4. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$. Let $\{A, B\}$ be a separated division of $\mathfrak{A}$. Then the points of $\mathfrak{B}$ for which the corresponding sections of A and B are not separated form a null set.*

Let A_e and B_e be outer associated sets of A and B respectively. Let $\mathfrak{D} = Dv\{A_e, B_e\}$. By *Lectures*, II, § 372, meas. $\mathfrak{D} = 0$. Let the sections of $\mathfrak{D}$ corresponding to points of $\mathfrak{B}$ be E. By *Lectures*, II, § 418,

$$\text{meas. } \mathfrak{D} = \int_{\mathfrak{B}} \overline{\overline{E}} = 0.$$

Therefore $\overline{\overline{E}}$ is a null function. Hence, by *Lectures*, II, § 402, 2, the points of $\mathfrak{B}$ for which $\overline{\overline{E}} > 0$ form a null set $\mathfrak{N}$. By *Lectures*, II, § 372, $\mathfrak{N}$ represents those points of $\mathfrak{B}$ for which the corresponding sections of A_e and B_e are not separated. But the points of $\mathfrak{B}$ for which the corresponding sections of A and B are not separated $\leqq \mathfrak{N}$, and form, therefore, a null set.

THEOREM 5. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be L-iterable with respect to $\mathfrak{B}$ which lies in r-way space $\mathfrak{R}_r$. Let $\{A, B\}$ be a separated division of $\mathfrak{A}$. Then A and B are also L-iterable with respect to their projections on $\mathfrak{R}_r$.*

Let $A = \mathfrak{B}_1 \cdot C$ and $B = \mathfrak{B}_2 \cdot D$. Let $\overline{\overline{C}}$ and $\overline{\overline{D}}$ be defined for each point of $\mathfrak{B}$ by setting them equal to zero for points of $\mathfrak{B}$ which are not points of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively. Let $\mathfrak{A}_e$, A_e, and B_e be outer associated sets of $\mathfrak{A}$, A, and B respectively, and $\mathfrak{C}_e$, C_e, and D_e be the respective sections of these sets.* By *Lectures*, II, § 376,

$$\overline{\overline{A}} + \overline{\overline{B}} = \overline{\overline{\mathfrak{A}}},$$

and by Theorem 4,

$$\overline{\overline{\mathfrak{C}}} = \overline{\overline{C}} + \overline{\overline{D}},$$

except possibly at the points of a null set in $\mathfrak{B}$. Therefore

$$\overline{\overline{A}} + \overline{\overline{B}} = \overline{\overline{\mathfrak{A}}} = \int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}} = \underline{\int}_{\mathfrak{B}} \overline{\overline{C}} + \overline{\int}_{\mathfrak{B}} \overline{\overline{D}} = \overline{\int}_{\mathfrak{B}} \overline{\overline{C}} + \underline{\int}_{\mathfrak{B}} \overline{\overline{D}}, \tag{1}$$

by *Lectures*, II, § 394, 1.

Now let $\mathfrak{A}_e$ be enclosed in an n-dimensional cube whose projection on $\mathfrak{R}_r$ is q. Let $\overline{C}_e = 0$ for points of q which are not points in the projection of A_e, and $\overline{D}_e = 0$ for points of q which are not points in the projection of B_e. Then, by *Lectures*, II, § 415,

$$\overline{\underline{\int}}_{\mathfrak{B}} \overline{\overline{C}} \leqq \int_q \overline{\overline{C}}_e = \overline{\overline{A}}, \qquad \overline{\underline{\int}}_{\mathfrak{B}} \overline{\overline{D}} \leqq \int_q \overline{\overline{D}}_e = \overline{\overline{B}}. \tag{2}$$

These with (1) and *Lectures*, II, § 401, give

$$\overline{\overline{A}} = \int_{\mathfrak{B}} \overline{\overline{C}} = \int_{\mathfrak{B}_1} \overline{\overline{C}}, \qquad \overline{\overline{B}} = \int_{\mathfrak{B}} \overline{\overline{D}} = \int_{\mathfrak{B}_2} \overline{\overline{D}}.$$

THEOREM 6. *Let A and B be parts of $\mathfrak{A}$ and $\overline{\overline{\mathfrak{A}}} = \overline{\overline{A}} + \overline{\overline{B}}$. Then A and B are separated parts of $\mathfrak{A}$.*

Let $\mathfrak{A}_e$, A_e, and B_e be outer associated sets of $\mathfrak{A}$, A, and B respectively. Evidently the sequences of standard ϵ_n-enclosures determining these sets may be so chosen that A_e and B_e contain no points not in $\mathfrak{A}_e$. Hence

$$\mathfrak{A} \leqq A_e + B_e - Dv\{A_e, B_e\} \leqq \mathfrak{A}_e.$$

Using *Lectures*, II, §§ 336 and 369, 2, we have

$$\overline{\overline{\mathfrak{A}}} = \overline{\text{meas.}}\,[A_e + B_e - Dv\{A_e, B_e\}].$$

But, by *Lectures*, II, §§ 369, 2 and 358, the sets in the square brackets are all measurable, and we have

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{A}} + \overline{\overline{B}} + \text{meas. } Dv\{A_e, B_e\}.$$

Hence, by hypothesis, meas. $Dv\{A_e, B_e\} = 0$, and consequently, by *Lectures*, II, § 372, A and B are separated sets and separated parts of $\mathfrak{A}$.

* Notice that $\mathfrak{C}_e$ is not necessarily an outer associated set of $\mathfrak{C}$ in the usual sense. A similar remark applies to C_e and D_e.

3. Separated Sets and Separated Functions in the Theory of *L*-Integration

Theorem 7. *Let $f(x_1, \cdots x_n) \geqq 0$ be L-integrable in $\mathfrak{A}$. Then the integrand set $\mathfrak{J}$ is a separated part of any enclosing ordinate set S.*

The ordinates of which S is composed are of constant length l. The function $l - f \geqq 0$ and is defined over $\mathfrak{A}$. It, in turn, defines an integrand set $\mathfrak{J}'$, which may be considered as being the set $S - \mathfrak{J}$. It is evident that

$$\bar{\bar{\mathfrak{J}}}' = \overline{\text{meas.}}\ (S - \mathfrak{J}), \qquad \int_{\mathfrak{A}} l = \bar{\bar{S}}.$$

These relations, with *Lectures,* II, §§ 394, 4 and 405, give

$$\bar{\bar{S}} - \bar{\bar{\mathfrak{J}}} = \int_{\mathfrak{A}} l - \int_{\mathfrak{A}} f = \int_{\mathfrak{A}} (l - f) = \bar{\bar{\mathfrak{J}}}' = \overline{\text{meas.}}\ (S - \mathfrak{J}).$$

Hence, by Theorem 6, $\mathfrak{J}$ is a separated part of S.

Theorem 8. *Let $f(x_1, \cdots x_n)$ be L-integrable in $\mathfrak{A}$. Then f is a separated function in $\mathfrak{A}$.*

First let $f \geqq 0$. Designate by S an enclosing ordinate set of the integrand set $\mathfrak{J}$. Let $\mathfrak{J}_\beta$ and S_β be the respective sets of $\mathfrak{J}$ and S whose $(n+1)$th coördinate is equal to β. Since, by Theorem 6, $\mathfrak{J}$ and $S - \mathfrak{J}$ are separated parts of S, the sections $\mathfrak{J}_\beta$ are separated parts of S_β, except possibly for a null set $\mathfrak{N}$ of values of β, by Theorem 4. Let b be an arbitrary but fixed value of β. Then there exists a sequence of values of β

$$\beta_1 < \beta_2 < \beta_3 \cdots \doteq b,$$

none of whose values lie in $\mathfrak{N}$. Hence each $\mathfrak{J}_{\beta_n}$ is a separated part of S_{β_n} and consequently $\mathfrak{A}_{\beta_n}$, which is the projection of $\mathfrak{J}_{\beta_n}$, is a separated part of $\mathfrak{A}$, which is the projection of S_{β_n}. As each $\mathfrak{A}_{\beta_{n+1}} \leqq \mathfrak{A}_{\beta_n}$, each point of $\mathfrak{A}_b$ lies in $\mathfrak{D} = Dv\{\mathfrak{A}_{\beta_n}\}$ so that

$$\mathfrak{A}_b \leqq \mathfrak{D}.$$

Also, as in the demonstration of *Lectures,* II, § 424

$$\mathfrak{A}_b \geqq \mathfrak{D}.$$

Hence

$$\mathfrak{A}_b = \mathfrak{D}.$$

But, by Theorem 1, $\mathfrak{D}$ is a separated part of $\mathfrak{A}$. Therefore $\mathfrak{A}_b$ is also a separated part of $\mathfrak{A}$. But $\mathfrak{A}_b$ is the set of points at which

$$f \geqq b.$$

Since b was arbitrary f is a separated function by definition.

The case where f is of unrestricted sign may now be proved as in the demonstration cited above.

I now state a theorem which is a further generalization of a theorem proved by W. H. Young.* I have proved the theorem, for a slightly less general case, in a former paper.† The proof as there given, with obvious changes, applies here.

THEOREM 9. *Let $f(x_1, \cdots x_n)$ be a separated function in $\mathfrak{A}$ such that $0 \leqq f < M$. Then f is L-integrable in $\mathfrak{A}$ and*

$$\int_{\mathfrak{A}} f = R \int_0^M \overline{\overline{\mathfrak{A}}}_\beta \, d\beta .$$

COROLLARY 1. *Let $f(x_1, \cdots x_n)$ be a separated function in $\mathfrak{A}$ such that $-N < f \leqq 0$. Then f is L-integrable in $\mathfrak{A}$ and*

$$\int_{\mathfrak{A}} f = -R \int_0^N \overline{\overline{\mathfrak{A}}}_\alpha \, d\alpha .$$

COROLLARY 2. *Let f be a separated function of both signs such that*

$$-N < f < M .$$

Then f is L-integrable in $\mathfrak{A}$ and

$$\int_{\mathfrak{A}} f = R \int_0^M \overline{\overline{\mathfrak{A}}}_\beta \, d\beta - R \int_0^N \overline{\overline{\mathfrak{A}}}_\alpha \, d\alpha .$$

THEOREM 10. *Let $f(x_1, \cdots x_n)$ be a separated function in $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$. Then f is a separated function in each $\mathfrak{C}$ belonging to a point of $\mathfrak{B}$, except possibly for the points of a null set $\mathfrak{N}$.*

Consider first the case where $f \geqq 0$. By Theorem 7, the integrand set $\mathfrak{J}$ is a separated part of any enclosing ordinate set S, and hence, by Theorem 4, each section $\mathfrak{J}(x)$, corresponding to a point x of $\mathfrak{B}$, is a separated part of $S(x)$, except possibly for the points of $\mathfrak{N}$. Consider some arbitrary but fixed $\mathfrak{C}(x)$, x not in $\mathfrak{N}$. Suppose that f is not a separated function in this $\mathfrak{C}$. The set B of values of β for which $\mathfrak{C}_\beta$ is not a separated part of $\mathfrak{C}$ is such that either, 1°, $\overline{\overline{B}} = 0$ or, 2°, $\overline{\overline{B}} > 0$. In the second case, by Theorem 4, $\mathfrak{J}(x)$ is not a separated part of $S(x)$ which is a contradiction. Consider now the first case. Let b be a point of B. Then we may choose a set of values of β

$$\beta_1 < \beta_2 < \beta_3 \cdots \doteq b$$

such that no β_n lies in B. But then, as in the proof of Theorem 8, $\mathfrak{C}_b$ is a separated part of $\mathfrak{C}$, which is a contradiction. Thus $\mathfrak{C}_\beta$ is a separated part of $\mathfrak{C}$ for each β, which proves our theorem in this case.

It is obvious that, if f is a separated function in $\mathfrak{A}$, it may be proved that

* *On upper and lower integration.* Proceedings of the London Mathematical Society, ser. 2, vol. 2 (1904), pp. 52–66. Also, *On the general theory of integration.* Philosophical Transactions, vol. 204 (1905), pp. 221–252.

† *On the continuity of a Lebesgue integral with respect to a parameter.* American Journal of Mathematics, vol. 36 (1914), p. 387.

f minus a constant, is also a separated function in $\mathfrak{A}$. Hence, if f is of both signs, we may choose a number C such that $f + C \geqq 0$. This function is then a separated function in $\mathfrak{A}$ and consequently in each $\mathfrak{C}$ except those corresponding to points of $\mathfrak{N}$. Hence f is also a separated function over these sections.

COROLLARY 1. *Let $f(x_1, \cdots x_n)$ be L-integrable in $\mathfrak{A}$. Then f is L-integrable in each $\mathfrak{C}$ belonging to a point of $\mathfrak{B}$, except possibly for the points of a null set $\mathfrak{N}$.*

THEOREM 11. *Let $f(x_1, \cdots x_n)$ be a separated function in $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ which is L-iterable with respect to $\mathfrak{B}$. Let $\mathfrak{b}$ denote those points of $\mathfrak{B}$ for which f is a separated function over the corresponding sections $\mathfrak{C}$. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{b}} \int_{\mathfrak{C}} f. \tag{1}$$

Let us first consider the case where $f \geqq 0$. As f is limited, we have

$$0 \leqq f < M.$$

As β ranges from 0 to M the field $\mathfrak{A}_\beta$ ranges from $\mathfrak{A}$ to zero. Let $Dv\{\mathfrak{A}_\beta, \mathfrak{C}\}$ be denoted by $\mathfrak{C}_\beta$. For points of $\mathfrak{B}$ which are not points of the projection of $\mathfrak{A}_\beta$ set $\overline{\overline{\mathfrak{C}}}_\beta = 0$. Then $\overline{\overline{\mathfrak{C}}}_\beta$ is defined over a field $A = \mathfrak{B} \cdot (0, M)$ and in turn defines a certain integrand set $\mathfrak{J}$, resting on A. Let S be an enclosing ordinate set of $\mathfrak{J}$. Denote $S - \mathfrak{J}$ by $\mathfrak{J}'$. Let the points of $\mathfrak{J}$, $\mathfrak{J}'$, and S corresponding to some β of the interval $(0, M)$, or to some point x of $\mathfrak{B}$, be $\mathfrak{J}(\beta)$, $\mathfrak{J}'(\beta)$, $S(\beta)$ and $\mathfrak{J}(x)$, $\mathfrak{J}'(x)$, $S(x)$ respectively. S consists then of a set of ordinates of length l erected on A, or it may be considered as a set of ordinates of length M erected on $B = \mathfrak{B} \cdot l$.* Thus the projection of any $S(\beta)$ on B is B itself.

Let us first show that for any β, arbitrary but fixed, $\mathfrak{J}(\beta)$ and $\mathfrak{J}'(\beta)$ are separated parts of $S(\beta)$. By hypothesis, $\mathfrak{A}_\beta$ is a separated part of $\mathfrak{A}$. Then, by the last step in the proof of Theorem 5, and using *Lectures,* II, § 405, we have

$$\overline{\overline{\mathfrak{A}}}_\beta = \int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}}_\beta = \overline{\overline{\mathfrak{J}}}(\beta). \tag{2}$$

But $\mathfrak{J}$ may also be considered as the integrand set of an auxiliary function F defined over $\mathfrak{J}(\beta = 0)$ and such that the set of points at which $F \geqq \beta$ is that part of $\mathfrak{J}(\beta = 0)$ which is the projection of $\mathfrak{J}(\beta)$ on B. Since $\mathfrak{J}(\beta)$ and $\mathfrak{J}'(\beta)$ are separated parts of $S(\beta)$ it is evident that F is a separated function in $\mathfrak{J}(\beta = 0)$. Hence, by Theorem 9 and *Lectures* II, § 405,

$$\int_{\mathfrak{J}(\beta=0)} F = \int_0^M \overline{\overline{\mathfrak{J}}}(\beta)\, d\beta = \overline{\overline{\mathfrak{J}}}, \tag{3}$$

and hence, by Theorem 7, $\mathfrak{J}$ and $\mathfrak{J}'$ are separated parts of S.

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* We may denote, for convenience, the points as well as the length of a section of B, corresponding to a point of $\mathfrak{B}$, by l.

For any point x of $\mathfrak{B}$, $\overline{\overline{\mathfrak{C}}}_\beta$ is a monotone decreasing function of β in $(0, M)$ and has, therefore, a Riemann integral, and

$$(4) \qquad R\int_0^M \overline{\overline{\mathfrak{C}}}_\beta \, d\beta = \int_0^M \overline{\overline{\mathfrak{C}}}_\beta \, d\beta = \overline{\overline{\mathfrak{J}}}(x),$$

by *Lectures*, II, §§ 381, 2 and 405. By *Lectures*, II, § 404,

$$\overline{\overline{S}} = l \cdot \overline{\overline{A}} = l \cdot (0, M) \cdot \overline{\overline{\mathfrak{B}}} = \overline{\overline{S}}(x) \cdot \overline{\overline{\mathfrak{B}}} = \int_{\mathfrak{B}} \overline{\overline{S}}(x).$$

Hence S is L-iterable with respect to $\mathfrak{B}$. But $\mathfrak{J}$ is a separated part of S and therefore, by Theorem 5, is also L-iterable. Hence

$$\int_{\mathfrak{B}} \overline{\overline{\mathfrak{J}}}(x) = \overline{\overline{\mathfrak{J}}}.$$

This with (4) gives

$$(5) \qquad \overline{\overline{\mathfrak{J}}} = \int_{\mathfrak{B}} R\int_0^M \overline{\overline{\mathfrak{C}}}_\beta \, d\beta = \int_{\mathfrak{b}} \int_{\mathfrak{C}} f,$$

by Theorems 9 and 10. Equations (5), (3), and (2) give

$$\int_0^M \overline{\overline{\mathfrak{A}}}_\beta = \int_{\mathfrak{b}} \int_{\mathfrak{C}} f.$$

Applying Theorem 9, we obtain (1), and our proof is complete for this case.

We may now prove the theorem for a function of unrestricted sign, in a similar manner, by using Theorem 9, Corollary 2, or by using a method similar to *Lectures*, II, § 422.

For any section $\mathfrak{C}$, corresponding to a point of $\mathfrak{b}$, the upper integral, the lower integral, and the integral of f over $\mathfrak{C}$ are identical. For points of $\mathfrak{B} - \mathfrak{b}$, f may not be integrable over the corresponding sections $\mathfrak{C}$. However, f has an upper and a lower integral over these sections. Since, by Theorem 10, $\mathfrak{B} - \mathfrak{b}$ is a null set we have from (1) and *Lectures*, II, § 380, 2:

THEOREM 12. *Let $f(x_1, \cdots x_n)$ be a separated function in $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ which is L-iterable with respect to $\mathfrak{B}$. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f.$$

THEOREM 13. *Let $f(x_1, \cdots x_n)$ be defined for the points of $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ and such that $0 < \mu \leqq f < M$. Let*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

Then f is a separated function in $\mathfrak{A}$ and $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is L-iterable with respect to $\mathfrak{B}$.

We have already proved in Theorem 8 that f is a separated function in $\mathfrak{A}$. It remains to show that

$$(1) \qquad \bar{\bar{\mathfrak{A}}} = \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}}.$$

Under Theorem 11 it is only the left hand equality in equations (2) that depends upon the fact that $\mathfrak{A}$ is L-iterable. Using the same notation as in the proof of that theorem, we have, from Theorem 9 and equations (5), (3), and (2) under Theorem 11,

$$(2) \qquad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f = \int_{\mathfrak{B}} R \int_0^M \bar{\bar{\mathfrak{C}}}_\beta \, d\beta = \bar{\bar{\mathfrak{J}}} = \int_0^M \bar{\bar{\mathfrak{J}}}(\beta) \, d\beta = \int_0^M \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}}_\beta \, d\beta.$$

Also by Theorem 9 and *Lectures*, II, § 381, 2,

$$\int_{\mathfrak{A}} f = R \int_0^M \bar{\bar{\mathfrak{A}}}_\beta \, d\beta = \int_0^M \bar{\bar{\mathfrak{A}}}_\beta \, d\beta.$$

By hypothesis, this relation with (2) gives

$$\int_0^M \bar{\bar{\mathfrak{A}}}_\beta \, d\beta = \int_0^M \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}}_\beta \, d\beta,$$

or

$$(3) \qquad \int_0^\mu \bar{\bar{\mathfrak{A}}} + \int_\mu^M \bar{\bar{\mathfrak{A}}}_\beta \, d\beta = \int_0^\mu \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}} + \int_\mu^M \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}}_\beta \, d\beta.$$

As in Theorem 5, equation (2),

$$(4) \qquad \bar{\bar{\mathfrak{A}}} \geqq \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}}, \qquad \bar{\bar{\mathfrak{A}}}_\beta \geqq \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}}_\beta.$$

Suppose that the inequality sign holds in the first relation of (4). Then (3) gives

$$\int_\mu^M \bar{\bar{\mathfrak{A}}}_\beta \, d\beta < \int_\mu^M \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}}_\beta \, d\beta.$$

But, by the second relation of (4)

$$\int_\mu^M \bar{\bar{\mathfrak{A}}}_\beta \, d\beta \geqq \int_\mu^M \int_{\mathfrak{B}} \bar{\bar{\mathfrak{C}}}_\beta \, d\beta,$$

which is a contradiction. Hence the equality sign holds in the first relation of (4), which proves (1).

Imagine the unit interval divided into two non-measurable Van Vleck sets,* v_1 and v_2. The following examples violate the conditions of Theorem 11 or of Theorem 13. As a consequence the conclusions of those theorems do not hold for these examples.

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* Loc. cit. See also *Lectures*, II, § 366.

Example 1. Let $f = 1$ for x in v_1, $0 \leqq y \leqq 1$,
$= 2$ for x in v_2, $0 \leqq y \leqq 1$.

$$\overline{\int_{\mathfrak{A}}} f = 2, \quad \underline{\int_{\mathfrak{A}}} f = 1.$$

In this example f is not a separated function in $\mathfrak{A}$.

Example 2. Let $f = 1$ for x in v_1, $0 \leqq y \leqq 1$,
$= 1$ for x in v_2, $1 \leqq y \leqq 2$.

$$\int_{\mathfrak{A}} f = 2, \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} f = 1.$$

Here $\mathfrak{A}$ is not L-iterable with respect to $\mathfrak{B}$.

Example 3. Let $f = 1$ for x in v_1, $0 \leqq y \leqq 1$,
$= 0$ for x in v_2, $1 \leqq y \leqq 2$.

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} = 1.$$

Here f is equal to zero at certain points. As a consequence $\mathfrak{A}$ is not L-iterable with respect to $\mathfrak{B}$.

Example 4. Let $f = 1$ for x in v_1, $0 \leqq y \leqq 1$,
$= 1$ for x in v_2, $\frac{1}{16} \leqq y \leqq 1\frac{1}{16}$,
$= -1$ for x in v_1, $1 < y \leqq 1\frac{1}{16}$,
$= -1$ for x in v_2, $1\frac{1}{16} < y \leqq 1\frac{1}{8}$.

$$\int_{\mathfrak{A}} f = 1\tfrac{1}{16} - \tfrac{1}{8} = \tfrac{15}{16}, \quad \int_{\mathfrak{B}} = 1 - \tfrac{1}{16} = \tfrac{15}{16}, \quad \int_{\mathfrak{B}} \int_{\mathfrak{C}} = \tfrac{15}{16}.$$

Hence

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}}; \quad \overline{\mathfrak{A}} = 1\tfrac{1}{8}, \quad \overline{\mathfrak{C}} = 1\tfrac{1}{16}, \quad \int_{\mathfrak{B}} \overline{\mathfrak{C}} = 1\tfrac{1}{16}.$$

Hence $\mathfrak{A}$ is not L-iterable with respect to $\mathfrak{B}$.

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